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The place of the adjoint representation in the Kronecker square of irreducible representations of simple Lie groups

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Abstract. The multiplicity of occurrence of the adjoint representation in the decomposition of the square of any finite-dimensional irreducible representation λ of any compact simple Lie group is shown to be equal to the number of non-vanishing components of the Dynkin label of λ. The resolution of this multiplicity into contributions to the symmetric and antisymmetric squares of λ is discussed, with complete results being found for all of the classical and some of the exceptional simple Lie groups, and partial results culminating in conjectures for the remaining exceptional groups.

1. Introduction

The calculation of the matrix elements of the generators of a compact simple Lie group $\mathcal{G}$ when acting on states spanning a particular finite-dimensional irreducible representation λ of $\mathcal{G}$ is a common problem in applications of Lie groups to physics (Judd 1963, Racah 1965). The generators of the group $\mathcal{G}$ transform as the adjoint irreducible representation θ of $\mathcal{G}$. A knowledge of the place of θ in the Kronecker square $\lambda \times \lambda$ of an arbitrary irreducible representation λ is of significance in evaluating matrix elements of group generators. It is well known, for example, that the eight-dimensional adjoint representation θ of the group $SU(3)$ occurs with multiplicity two in the Kronecker square $\theta \times \theta$. This is an important feature of the quark model (Gell-Mann and Ne'eman 1964, Pais 1966).

Within this model couplings are further influenced by the fact that in the decomposition of the Kronecker square $\theta \times \theta$ into its symmetric and antisymmetric parts, $\theta \otimes \{2\}$ and $\theta \otimes \{1^2\}$ respectively, the adjoint appears with multiplicity one in each part. However, this equality of multiplicities of the adjoint θ in the symmetric and antisymmetric parts of the Kronecker square of an arbitrary finite-dimensional irreducible representation λ of an arbitrary compact simple Lie group $\mathcal{G}$, while not being untypical in the case of $SU(k+1)$, is the exception rather than the rule for the other compact simple Lie groups. In many instances the occurrence of θ is confined to just one or other of these two parts.

Indeed this work arose from observations based on the resolution of Kronecker squares of irreducible representations of the compact simple Lie groups into their symmetric and antisymmetric parts using SCHUR†. The results obtained in this way led us to pose the question: 'When does the adjoint θ occur only in the symmetric or only in the antisymmetric

† SCHUR is an interactive C package for calculating properties of Lie groups and symmetric functions. Distributed by: S Christensen, PO Box 16175, Chapel Hill, NC 27516, USA. E-mail: steve@scm.vnet.net. A detailed description can be seen by WEB users at http://scm.vnet.net/Christensen.html.

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part of the Kronecker square of a given irreducible representation λ of $\mathcal{G}$?'. Herein we give an answer to this question in the form of propositions covering $SU(k+1)$, $SO(2k+1)$, $Sp(2k)$, $SO(2k)$, E_7 and G_2, and conjectures covering the remaining cases, E_6, E_8 and F_4. At the same time we provide explicit formulae for the relevant multiplicities of occurrence of the adjoint. Throughout we take advantage of the fact that the representation theory of each compact simple Lie group $\mathcal{G}$ is determined by that of the corresponding complex simple Lie algebra $\boldsymbol{g}$.

In section 2 the notation for irreducible representations and their characters is established, while in section 3 some preliminary results are presented which follow from Weyl's character formula and the algebra of Schur functions. The multiplicity of occurrence of the adjoint irreducible representation θ in the Kronecker square $\lambda \times \lambda$ is determined in section 4 for all irreducible representations λ of each compact simple Lie group $\mathcal{G}$. In particular it is shown that this multiplicity is non-vanishing if and only if λ is both non-trivial and selfcontragredient. The irreducible representations of $\mathcal{G}$ which are selfcontragredient are identified in section 5, and further classified as either orthogonal or symplectic. This distinction is particularly important in determining the multiplicities of occurrence of the adjoint irreducible representation θ in the symmetric and antisymmetric squares, $\lambda \otimes \{2\}$ and $\lambda \otimes \{1^2\}$, respectively. This determination is carried out for $SU(k+1)$ in section 6. The remaining classical compact simple Lie groups , $SO(2k+1)$, $Sp(2k)$ and $SO(2k)$ are dealt with in section 7, while the exceptional simple Lie groups are covered in section 8.

2. Natural and Dynkin labels for irreducible representations

Each compact simple Lie group $\mathcal{G}$ is associated with the unique compact real form of a complex simple Lie algebra $\boldsymbol{g}$, and their finite-dimensional irreducible representations λ are in one-to-one correspondence. Each of these irreducible representations is defined, up to equivalence, by its highest weight λ. This highest weight λ can itself be specified in more than one way, using for example, either Dynkin labels (Dynkin 1957, McKay and Patera 1981) or natural labels involving partitions (Wybourne and Bowick 1977, King and Al-Qubanchi 1981, Black *et al* 1983). Of these the former have the advantage of allowing all compact simple Lie groups to be dealt with in a uniform manner, while the latter are particularly useful in dealing with the four infinite series of compact classical simple Lie groups in a rank independent way.

Let $\boldsymbol{g}$ be the complex simple Lie algebra associated with the compact simple Lie group $\mathcal{G}$. Let $\boldsymbol{h}$ be the Cartan subalgebra of $\boldsymbol{g}$ and let $\boldsymbol{h}^*$ be the dual of $\boldsymbol{h}$. Let Δ and Π denote the sets of roots and simple roots, respectively, of $\boldsymbol{g}$. For each $\alpha_i \in \Pi$ let $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$, where $(\cdot,\cdot)$ signifies the inner product on $\boldsymbol{h}^*$.

Each finite-dimensional irreducible representation λ of $\boldsymbol{g}$ corresponds to a highest weight module V^λ and is specified up to equivalence by its highest weight λ. If $\boldsymbol{g}$ has rank k then the corresponding Dynkin labels are given by $a_i = (\lambda, \alpha_i^\vee)$ for $i = 1, \ldots, k$. In the basis of fundamental weights (Bremner *et al* 1985) ω_i with $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ for $i, j = 1, 2, \ldots, k$, we have $\lambda = \sum_{i=1}^k a_i \omega_i$, and it is convenient to write $\lambda = (a_1, a_2, \ldots, a_k))$. Of necessity λ is dominant with each component a_i a non-negative integer. A quantity of particular interest in what follows is the number of non-vanishing Dynkin labels a_i, which we refer to as the breadth of λ and denote by $b(\lambda)$.

Alternatively we can make use of partitions to specify irreducible representations of $\mathcal{G}$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ signify a partition of length $\ell(\lambda)$ and weight w_λ. The parts λ_i of λ for $i = 1, 2, \ldots, \ell$ are positive integers, with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{\ell(\lambda)} > 0$, whose sum is w_λ. Let the number of distinct parts of λ be $d(\lambda)$, a key parameter in what follows.

All the finite-dimensional irreducible representations of the classical and exceptional Lie groups $\mathcal{G}$ may be labelled by means of partitions (Wybourne and Bowick 1977, King and Al-Qubanchi 1981, Black *et al* 1983). This labelling is natural in the sense that it indicates the tensor or spinor structure of corresponding modules of $\mathcal{G}$. The precise connection between Dynkin labels and natural labels has been spelled out in detail by King and Al-Qubanchi (1981), with a further refinement in the labelling being provided by Black *et al* (1983). We refer the reader to these papers for much of the notation employed here. This notation is such that the representation labels are to be identified with characters of the corresponding representations of both $\mathcal{G}$ and $\boldsymbol{g}$. In this way no distinction need, nor will be made between characters of $\mathcal{G}$ and $\boldsymbol{g}$, although in general we choose to talk about characters or irreducible representations of $\mathcal{G}$.

3. Preliminary lemmas

If we work in terms of natural labels involving partitions then the connection with Schur functions provides a uniform framework in which to describe the decomposition of Kronecker products of all irreducible representations of all the classical compact simple Lie groups, independent of their rank (Black *et al* 1983). At the heart of these methods lies the fact that the characters of the irreducible representations λ can all be expressed in terms of Schur functions whose products and quotients are described by the Littlewood–Richardson rule (Littlewood 1950, Macdonald 1995). As usual we denote Schur function products and quotients by $\cdot$ and $/$, while symmetrized products or plethysms are denoted by $\otimes$. Recalling our definition of $d(\lambda)$ as the number of distinct parts of the partition λ, the following two lemmas then follow trivially from the properties of Schur functions:

Lemma 3.1. $\lambda \cdot 1$ and $\lambda/1$ contain $d(\lambda)+1$ and $d(\lambda)$ distinct terms, respectively.

Lemma 3.2. $(\lambda/1)\cdot 1$ contains λ with multiplicity $d(\lambda)$.

On the other hand if we work in terms of formal exponentials the key tool at our disposal is Weyl's character formula (Humphreys 1972):

$$ch\ V^\lambda = \frac{\sum_{w\in W} \epsilon(w)\, \mathrm{e}^{w(\lambda+\rho)}}{\sum_{w\in W} \epsilon(w)\, \mathrm{e}^{w(\rho)}} \tag{3.1}$$

where the summations are over all elements w of the Weyl group W of $\boldsymbol{g}$, $\epsilon(w)$ is the parity or signature of w, and ρ is half the sum of the positive roots of $\boldsymbol{g}$. The Weyl group W is generated by the reflections r_i, with $i = 1, \ldots, k$, whose action on any weight μ is defined by $r_i(\mu) = \mu - (\mu, \alpha_i^\vee)\alpha_i$. If $w = r_{i_1} r_{i_2} \cdots r_{i_m}$ then $\epsilon(w) = (-1)^m$. The expansion of (3.1) in the form:

$$ch\ V^\lambda = \sum_{\mu\in \boldsymbol{h}^*} m_\mu^\lambda\, \mathrm{e}^\mu \tag{3.2}$$

serves to define the weights μ of V^λ and their multiplicities m_μ^λ.

Weyl's character formula (3.1) with λ replaced by μ also serves to define formal characters $ch\ V^\mu$ for any weight μ of any complex simple Lie algebra $\boldsymbol{g}$ of rank k whose Weyl group is W. For such characters it is easy to derive from (3.1):

Lemma 3.3. $ch\ V^\mu = \epsilon(w) ch\ V^{w(\mu+\rho)-\rho}$ for any $w \in W$.

Lemma 3.4. $ch\ V^{\mu} = 0$ if $(\mu, \alpha_i^{\vee}) = -1$ for any $i \in \{1, \ldots, k\}$.

The adjoint irreducible representation θ of each compact simple Lie group $\mathcal{G}$ is identified in table 1 in terms of both natural and Dynkin labels. This irreducible representation θ is characterized by the fact that its weights coincide with the set Δ of all roots α of $\boldsymbol{g}$, each with multiplicity one, together with the zero vector with multiplicity equal to the rank k of $\boldsymbol{g}$. The expansion (3.2) of Weyl's character formula for the adjoint irreducible representation θ therefore takes the form:

$$ch\ V^{\theta} = k\ 1 + \sum_{\alpha\in\Delta} \mathrm{e}^{\alpha} = k\ 1 + \sum_{\alpha\in\Delta} \mathrm{e}^{w(\alpha)} \qquad \text{for any } w \in W \tag{3.3}$$

where the last step depends on the fact that Δ is invariant under the action of the Weyl group W.

Table 1. Natural and Dynkin labels for the adjoint irreducible representation θ of the compact simple Lie groups $\mathcal{G}$ of rank k.

$\mathcal{G}$	Adjoint irreducible representation θ	Dynkin label $((a_1, a_2, \ldots, a_k))$
$SU(k+1) = A_k$	$\{\bar{1}; 1\} = \{21^{k-1}\}$	$((1000\cdots000))$
$SO(2k+1) = B_k$	$[1^2]$	$((010\cdots000))$
$Sp(2k) = C_k$	$\langle 2\rangle$	$((200\cdots000))$
$SO(2k) = D_k$	$[1^2]$	$((010\cdots000))$
E_6	(2;0)	((000001))
E_7	(21^6)	((1000000))
E_8	(21^7)	((10000000))
F_4	(1^2)	((1000))
G_2	(21)	((10))

Turning, more generally, to arbitrary representations of $\mathcal{G}$, the symmetric, bilinear, inner product $\langle\cdot,\cdot\rangle$ on the set of characters of equivalence classes of irreducible representations of $\mathcal{G}$ is such that

$$ch\ V^{\lambda\times\mu} = ch\ V^{\lambda} ch\ V^{\mu} = \sum_{\nu} \langle\lambda\times\mu, \nu\rangle ch\ V^{\nu} \tag{3.4}$$

where $\langle\lambda\times\mu, \nu\rangle$ is the multiplicity of occurrence of the irreducible representation ν in the Kronecker product $\lambda\times\mu$, and the sum is taken over all irreducible representations ν of $\mathcal{G}$.

Here we are particularly interested in the decomposition of the Kronecker square $\lambda\times\lambda$ of an irreducible representation λ into its symmetric and antisymmmetric parts $\lambda\otimes\{2\}$ and $\lambda\otimes\{1^2\}$. If A and B denote arbitrary linear combinations of irreducible representations of a compact simple Lie group $\mathcal{G}$, then the algebra of plethysms is such that (Littlewood 1950):

$$A\times A = A\otimes\{2\} + A\otimes\{1^2\} \tag{3.5}$$

$$(A+B)\otimes\{2\} = A\otimes\{2\} + A\times B + B\otimes\{2\} \tag{3.6a}$$

$$(A+B)\otimes\{1^2\} = A\otimes\{1^2\} + A\times B + B\otimes\{1^2\} \tag{3.6b}$$

$$(A\times B)\otimes\{2\} = (A\otimes\{2\})\times(B\otimes\{2\}) + (A\otimes\{1^2\})\times(B\otimes\{1^2\}) \tag{3.7a}$$

$$(A\times B)\otimes\{1^2\} = (A\otimes\{2\})\times(B\otimes\{1^2\}) + (A\otimes\{1^2\})\times(B\otimes\{2\}). \tag{3.7b}$$

It follows from (3.5) that the multiplicities of occurrence of the adjoint irreducible representation θ in the Kronecker square, the symmetrized and the antisymmetrized squares of an arbitrary irreducible representation λ of a simple Lie group $\mathcal{G}$ are such that:

$$\langle\lambda\times\lambda, \theta\rangle = \langle\lambda\otimes\{2\}, \theta\rangle + \langle\lambda\otimes\{1^2\}, \theta\rangle. \tag{3.8}$$

To every irreducible representation λ of a compact simple Lie group $\mathcal{G}$ there corresponds a contragredient or dual irreducible representation $\bar{\lambda}$ such that if η denotes the trivial, identity, one-dimensional irreducible representation of $\mathcal{G}$, then

$$\langle \lambda \times \mu, \eta \rangle = \delta_{\mu\bar{\lambda}}. \tag{3.9}$$

An irreducible representation λ is said to be selfcontragredient if $\bar{\lambda} = \lambda$.

The contragredient $\bar{\lambda}$ of λ is characterized by the fact that the weights of $\bar{\lambda}$ are equal to the weights of λ taken with opposite sign, but no change in multiplicity (Mal'cev 1962). Clearly the adjoint irreducible representation θ is selfcontragredient since its only non-vanishing weights are the roots $\alpha \in \Delta$, each having multiplicity one, and $\alpha \in \Delta$ implies $-\alpha \in \Delta$.

More generally, all the irreducible representations λ of $SO(2k+1)$, $Sp(2k)$, $SO(2k)$ with k even, E_7, E_8, F_4 and G_2 are selfcontragredient so that $\bar{\lambda} = \lambda$. In the case of $SU(k+1)$, $SO(2k)$ with k odd, and E_6 the irreducible representations $\bar{\lambda}$ contragredient to λ are given in table 2 in terms of both natural and Dynkin labels.

Table 2. The irreducible representation $\bar{\lambda}$ contragredient to each irreducible representation $\lambda = ((a_1, \ldots, a_k))$ of $SU(k+1)$, $SO(2k)$ with k odd, and E_6.

$\mathcal{G}$	λ	$\bar{\lambda}$	$\overline{((a_1, a_2, \ldots, a_{k-1}, a_k))}$
$SU(k+1) = A_k$	$\{\lambda\}$	$\{\bar{\lambda}\}$ with $\bar{\lambda}_1 = \lambda_1$ and $\bar{\lambda}_i = \lambda_1 - \lambda_{k-i+2}$ for $i = 2, \ldots, k$	$((a_k, a_{k-1}, \ldots, a_2, a_1))$
$SO(2k)$	$[\lambda]_\pm$	$[\lambda]_\mp$	$((a_1, \ldots, a_{k-2}, a_k, a_{k-1}))$
with k odd	$[\Delta; \lambda]_\pm$	$[\Delta; \lambda]_\mp$	
E_6	(λ_0, λ)	$(\lambda_0, \bar{\lambda})$ with $\bar{\lambda}_1 = \lambda_1$ and $\bar{\lambda}_i = \lambda_1 - \lambda_{7-i}$ for $i = 2, \ldots, 5$	$((a_5, a_4, a_3, a_2, a_1, a_6))$

An alternative characterization of selfcontragredient irreducible representations of $\mathcal{G}$ is provided by the following proposition which follows immediately from (3.9):

Proposition 3.5. The irreducible representation λ of a compact simple Lie group $\mathcal{G}$ is selfcontragredient if and only if $\langle \lambda \times \lambda, \eta \rangle = 1$, where η is the identity representation of $\mathcal{G}$.

If an irreducible representation λ of a compact simple Lie group $\mathcal{G}$ is not selfcontragredient then its character is complex. In the case of a selfcontragredient irreducible representation λ the character is real, and the representation matrices themselves are either orthogonal or symplectic according as their Kronecker square supports a symmetric or an antisymmetric bilinear form. More precisely, bearing in mind (3.5) and Proposition 3.5 which imply that there are indeed only two possibilities, we have:

Proposition 3.6. A selfcontragredient irreducible representation λ of a compact simple Lie group $\mathcal{G}$ is orthogonal if $\langle \lambda \otimes \{2\}, \eta \rangle = 1$ and is symplectic if $\langle \lambda \otimes \{1^2\}, \eta \rangle = 1$, where η is the identity irreducible representation of $\mathcal{G}$.

The identification of orthogonal and symplectic irreducible representations for all the compact simple Lie groups is well known (Dynkin 1957, Mal'cev 1962, Mehta 1966, Mehta and Srivastava 1966, Butler and King 1974, McKay and Patera 1981). The data are summarized in tables 3 and 4 in terms of natural and Dynkin labels, respectively.

Table 3. Orthogonal and symplectic irreducible representations λ of the Lie group $\mathcal{G}$ of rank k in terms of natural labels.

$\mathcal{G}$	Selfcontragredient λ	Orthogonal	Symplectic
$SU(k+1)$	$\{\lambda\}$ with $\lambda_i = \lambda_1 - \lambda_{k-i+2}$	$k = 0, 2, 3(\mathrm{mod}\,4)$	
	for $2 \leqslant i \leqslant [(k+1)/2]$	$k = 1(\mathrm{mod}\,4), \lambda_1 = 0(\mathrm{mod}\,2)$	$k = 1(\mathrm{mod}\,4), \lambda_1 = 1(\mathrm{mod}\,2)$
$SO(2k+1)$	$[\lambda]$	all	
	$[\Delta; \lambda]$	$k = 0, 3(\mathrm{mod}\,4)$	$k = 1, 2(\mathrm{mod}\,4)$
$Sp(2k)$	$\langle\lambda\rangle$	$w_\lambda = 0(\mathrm{mod}\,2)$	$w_\lambda = 1(\mathrm{mod}\,2)$
$SO(2k)$	$[\lambda]$	all	
	$[\lambda]_\pm \quad k = 0(\mathrm{mod}\,2)$	all selfcontragredient	
	$[\Delta; \lambda]_\pm \quad k = 0(\mathrm{mod}\,2)$	$k = 0(\mathrm{mod}\,4)$	$k = 2(\mathrm{mod}\,4)$
G_2	(λ)	all	
F_4	(λ)	all	
	$(\Delta; \lambda)$	all	
E_6	$(\lambda_0; \lambda)$ with	all selfcontragredient	
	$\lambda_1 = \lambda_2 + \lambda_5 = \lambda_3 + \lambda_4$		
E_7	(λ)	$w_\lambda = 0(\mathrm{mod}\,4)$	$w_\lambda = 2(\mathrm{mod}\,4)$
E_8	(λ)	all	

Table 4. Orthogonal and symplectic irreducible representations $\lambda = ((a_1, \ldots, a_k))$ of the Lie group $\mathcal{G}$ of rank k in terms of Dynkin labels.

$\mathcal{G}$	Selfcontragredient λ		Orthogonal	Symplectic
$SU(k+1)$	$a_i = a_{k-i+1}$		$k = 0, 2, 3(\mathrm{mod}\,4)$	
	for $i = 1, \ldots, [k/2]$		$a_{[(k+1)/2]} = 0(\mathrm{mod}\,2)$ $k = 1(\mathrm{mod}\,4)$	$a_{[(k+1)/2]} = 1(\mathrm{mod}\,2)$ $k = 1(\mathrm{mod}\,4)$
$SO(2k+1)$	all		$a_k = 0(\mathrm{mod}\,2)$ all k	
			$a_k = 1(\mathrm{mod}\,2)$ $k = 0, 3(\mathrm{mod}\,4)$	$a_k = 1(\mathrm{mod}\,2)$ $k = 1, 2(\mathrm{mod}\,4)$
$Sp(2k)$	all		$a_1 + a_3 + a_5 + \cdots = 0(\mathrm{mod}\,2)$	$a_1 + a_3 + a_5 + \cdots = 1(\mathrm{mod}\,2)$
$SO(2k)$	$k = 0(\mathrm{mod}\,4)$	all	all	
	$k = 1, 3(\mathrm{mod}\,4)$	$a_{k-1} = a_k$	all selfcontragredient	
	$k = 2(\mathrm{mod}\,4)$	all	$a_{k-1} + a_k = 0(\mathrm{mod}\,2)$	$a_{k-1} + a_k = 1(\mathrm{mod}\,2)$
G_2	all		all	
F_4	all		all	
E_6	$a_1 = a_5, a_2 = a_4$		all selfcontragredient	
E_7	all		$a_4 + a_6 + a_7 = 0(\mathrm{mod}\,2)$	$a_4 + a_6 + a_7 = 1(\mathrm{mod}\,2)$
E_8	all		all	

The following lemmas regarding arbitrary irreducible representations λ, μ and ν of a simple Lie group $\mathcal{G}$ whose adjoint irreducible representation is θ may be readily derived and are of considerable use in what follows:

Lemma 3.7. $\langle\lambda \times \mu, \nu\rangle = \langle\lambda \times \overline{\nu}, \overline{\mu}\rangle$.

Lemma 3.8. $\langle\lambda \times \lambda, \theta\rangle = \langle\lambda \times \theta, \overline{\lambda}\rangle$.

For the record it should also be noted that an inspection of the tabulation of roots $\alpha \in \Delta$ of each simple Lie group $\mathcal{G}$ in the basis of fundamental weights ω_i given by Bremner *et al* (1985) reveals that for the highest weight λ of any irreducible representation of $\mathcal{G}$:

Lemma 3.9. $\lambda + \alpha \neq \overline{\lambda}$ for any $\alpha \in \Delta$.

4. Kronecker squares

We are interested in the multiplicity of occurrence of the adjoint irreducible representation θ in the Kronecker square $\lambda \times \lambda$ of each finite-dimensional irreducible representation λ of each compact simple Lie group $\mathcal{G}$. From lemma 3.8 the required multiplicity is that of $\bar{\lambda}$ in the Kronecker product $\lambda \times \theta$.

However, from (3.1) and (3.3)

$$
\begin{aligned}
ch\, V^{\lambda} ch\, V^{\theta} &= k ch\, V^{\lambda} + \sum_{\alpha \in \Delta} \frac{\sum_{w \in W} \epsilon(w)\, \mathrm{e}^{w(\lambda+\rho+\alpha)}}{\sum_{w \in W} \epsilon(w)\, \mathrm{e}^{w(\rho)}} \\
&= k ch\, V^{\lambda} + \sum_{\alpha \in \Delta} ch\, V^{\lambda+\alpha} \qquad (4.1)
\end{aligned}
$$

where $\lambda + \alpha$ may or may not be dominant. There are three cases to consider:

(i) if $\lambda + \alpha$ is dominant there is no problem since $ch\, V^{\lambda+\alpha}$ is standard;

(ii) if $\lambda+\alpha$ is not dominant but there exists any $i \in \{1, \ldots, k\}$ such that $(\lambda+\alpha, \alpha_i^\vee) = -1$ then $ch\, V^{\lambda+\alpha} = 0$ by virtue of lemma 3.4. In such a case we say that $\lambda + \alpha$ is null;

(iii) if $\lambda + \alpha$ is not dominant and not null then there exists $i \in \{1, \ldots, k\}$ such that $(\lambda+\alpha, \alpha_i^\vee) \leqslant -2$ and the corresponding contribution $ch\, V^{\lambda+\alpha}$ to (4.1) must be standardized through the use of lemma 3.3.

For example, if $\alpha = -\alpha_i$ for some $i \in \{1, \ldots, k\}$ then $(\lambda+\alpha, \alpha_i^\vee) = (\lambda-\alpha_i, \alpha_i^\vee) = a_i - 2$ and $(\lambda + \alpha, \alpha_j^\vee) = (\lambda - \alpha_i, \alpha_j^\vee) = a_j - (\alpha_i, \alpha_j^\vee) \geqslant a_j \geqslant 0$ for all $j \neq i$. It follows that $\lambda - \alpha_i$ is dominant if $a_i \geqslant 2$ and null if $a_i = 1$, but is non-dominant and non-null if $a_i = 0$. However, if $a_i = 0$ we have

$$
r_i(\lambda - \alpha_i + \rho) - \rho = \lambda - \alpha_i - (\lambda - \alpha_i + \rho, \alpha_i^\vee)\alpha_i = \lambda - a_i\alpha_i = \lambda \qquad (4.2)
$$

since $(\rho, \alpha_i^\vee) = 1$ for all $i - 1, 2, \ldots, k$. It then follows from lemma 3.3 with $w = r_i$ that

$$
ch\, V^{\lambda-\alpha_i} = -ch\, V^{\lambda} \qquad \text{if} \qquad a_i = 0. \qquad (4.3)
$$

More generally, in case (iii) since λ is dominant with $(\lambda, \alpha_i^\vee) = a_i \geqslant 0$ for all $i \in \{1, \ldots, k\}$, there must exist $i \in \{1, \ldots, k\}$ such that $(\alpha, \alpha_i^\vee) = -p$ with $p \geqslant a_i + 2 \geqslant 2$. However, $(\alpha, \beta^\vee) \in \{0, \pm 1, \pm 2, \pm 3\}$ for all $\beta \in \Delta$ (Humphreys 1972), so there are just two possibilities, namely $p = 2$ and $p = 3$, with the latter only occurring in the case $\mathcal{G} = G_2$. Moreover, an examination of the tables of Bremner *et al* (1985) shows that for any given $\alpha \in \Delta$ if $(\alpha, \alpha_i^\vee) = -p$ for some $i \in \{1, \ldots, k\}$ with $p = 2$ or 3 then that value of i is unique. We can distinguish between the two possibilities: (a) $\alpha = -\alpha_i$ and (b) $\alpha \neq -\alpha_i$. The first of these has already been dealt with. In fact it is the only possibility for each of the simply laced algebras A_k, D_k, E_6, E_7 and E_8.

Turning to case (b), if $\alpha \neq -\alpha_i$ then α is not a multiple of α_i since the only other possibility is $\alpha = +\alpha_i$ in which case $(\alpha, \alpha_i^\vee) = -p = 2$, in contradiction with the requirement that $p = 2$ or 3. However, if α is not a multiple of α_i, then

$$
r_i(\alpha) = \alpha - (\alpha, \alpha_i^\vee)\alpha_i = \alpha + p\alpha_i \qquad (4.4)
$$

and there necessarily exists a chain of roots $\alpha + r\alpha_i$ with $r = 0, 1, \ldots, p$ (Humphreys 1972). In addition,

$$
r_i(\lambda + \alpha + \rho) - \rho = \lambda + \alpha - (\lambda + \alpha + \rho, \alpha_i^\vee)\alpha_i = \lambda + \alpha - (a_i - p + 1)\alpha_i = \lambda + \beta \qquad (4.5)
$$

where $\beta = \alpha + q\alpha_i$ with $q = p - 1 - a_i$. Recalling that $p \geqslant a_i + 2$ and $a_i \geqslant 0$ it follows that $1 \leqslant q < p$, so that β is necessarily a root. Thus, from (4.5) and lemma 3.3

$$
ch\, V^{\lambda+\alpha} = -ch\, V^{\lambda+\beta} \qquad \text{with} \qquad \beta = \alpha + q\alpha_i \in \Delta \qquad (4.6)
$$

and we have a cancellation of contributions to (4.1) of the terms arising from $\alpha \in \Delta$ and $\beta \in \Delta$. To be sure that this is the end of the story we have to be sure that all the β obtained by means of (4.5) from different α are distinct.

It is to be noted that having identified all relevant α and i from the tables of Bremner *et al* (1985), then $\beta = \alpha + q\alpha_i$ with q restricted to be 1 or 2. In fact if $p = 2$ then the condition $p \geqslant a_i + 2 \geqslant 2$ implies that $a_i = 0$ so that $q = 1$. For the non-simply laced algebras B_k, C_k, F_4 and G_2, this covers all possibilities except in the case of G_2 for which it is necessary to consider $p = 3$. In this case we have either $a_i = 1$ so that $q = 1$ as before, or $a_i = 0$ so that $q = 2$. Again consulting the tables of Bremner *et al* (1985) to obtain the list of roots $\beta = \alpha + q\alpha_i$, it is indeed found in every case that the β arising from different α are distinct. Moreover in every case $(\beta, \alpha_j^\vee) \in \{0, \pm 1\}$ for all $j \in \{1, \ldots, k\}$ so that $\lambda + \beta$ is either dominant or null. In all cases we therefore have the cancellation of contributions to (4.1) implied by (4.6), although in some cases these contributions are in fact null.

Applying (4.3) and (4.6) to (4.1), together with the observations made regarding cases (i) and (ii), we have the following:

Proposition 4.1. Let λ be any finite-dimensional irreducible representation of a compact simple Lie group $\mathcal{G}$ whose adjoint irreducible representation is θ, and let $b(\lambda)$ be the number of non-vanishing components of the Dynkin label $\lambda = ((a_1, \ldots, a_k))$. Then

$$ch\, V^\lambda ch\, V^\theta = b(\lambda) ch\, V^\lambda + \sum_{\alpha \in \Delta^\lambda} ch\, V^{\lambda+\alpha}$$

where Δ^λ is the set of roots $\alpha \in \Delta$ such that $\lambda + \alpha$ is dominant, and there exists no $\beta \in \Delta$ such that $r_i(\lambda + \beta + \rho) = \lambda + \alpha + \rho$ for any $i \in \{1, \ldots, k\}$.

Thanks to lemmas 3.8 and 3.9 this immediately gives us one of our key results:

Proposition 4.2. For any compact simple Lie group $\mathcal{G}$, the multiplicity of occurrence of the adjoint irreducible representation θ in the Kronecker square of any finite-dimensional irreducible representation λ is non-zero if and only if λ is selfcontragredient. If λ is selfcontragredient this multiplicity is given by $\langle \lambda \times \lambda, \theta \rangle = b(\lambda)$, where $b(\lambda)$ is the number of non-vanishing components of the Dynkin label $\lambda = ((a_1, \ldots, a_k))$.

The above proposition was stated earlier by Elashvili (1992) but he provided only a partial proof of its validity. In particular he gave no justification of the fact that $\langle \lambda \times \lambda, \theta \rangle$ is equal to $b(\lambda)$.

5. Symmetrized Kronecker squares for $SU(k + 1)$

While the results of section 4 embodied in propositions 4.1 and 4.2 are completely general in the sense that they apply to any compact simple Lie group $\mathcal{G}$, it is worth pointing out that the same results may be derived rather easily using Schur function methods in the case of $SU(k + 1)$.

In terms of natural labels an arbitrary finite-dimensional irreducible representation is denoted by $\{\lambda\}$, where λ is a partition of length $\ell(\lambda) \leqslant k$, and the adjoint irreducible representation is given by $\theta = \{21^{k-1}\}$. The Kronecker product $\{\lambda\} \times \{21^{k-1}\}$ may be evaluated quite readily through the use of the Littlewood–Richardson rule (Littlewood 1950, Macdonald 1995). However, it is advantageous to make use of the freedom associated with the constraint $x_1 x_2 \cdots x_{k+1} = 1$, which applies to all Schur functions corresponding to characters of $SU(k+1)$, to express $\theta = \{21^{k-1}\}$ in the form $\{\bar{1}; 1\}$. This composite partition notation emphasizes the fact that the adjoint irreducible representation $\theta = \{\bar{1}; 1\}$ of $SU(k+1)$ appears in the decomposition of the Kronecker product of the defining, fundamental irreducible representation $\omega_1 = \{1\}$ and its contragredient $\omega_k = \{1^k\} = \{\bar{1}\}$. This product takes the form:

$$\{\bar{1}\} \times \{1\} = \{\bar{1}; 1\} + \{0\}. \tag{5.1}$$

With this notation (Black *et al* 1983), the Kronecker product of an arbitrary irreducible representation $\{\lambda\}$ of $SU(k+1)$ with the adjoint irreducible representation decomposes in accordance with the formula

$$\{\lambda\} \times \{\bar{1}; 1\} = \{\bar{1}; \lambda \cdot 1\} + \{(\lambda/1) \cdot 1\}. \tag{5.2}$$

In general modification rules (Black *et al* 1983) may have to be brought into play. However, since $\ell(\lambda) \leqslant k$ all terms in (5.2) are standard except the term $\{\bar{1}; \lambda, 1\}$ if $\ell(\lambda) = k$, and in such a case $\{\bar{1}; \lambda, 1\}$ is identically zero.

It then follows from lemma 3.2 that the multiplicity of $\{\lambda\}$ in the Kronecker product (5.2) is given by:

$$\langle \{\lambda\} \times \theta, \{\lambda\} \rangle = d(\lambda). \tag{5.3}$$

Lemma 3.5 then implies that if $\{\lambda\}$ is selfcontragredient then

$$\langle \{\lambda\} \times \{\lambda\}, \theta \rangle = d(\lambda). \tag{5.4}$$

That this is in agreement with proposition 4.2 may be seen by noting (King and Al-Qubanchi 1981) that $a_i = \lambda_i - \lambda_{i+1}$ for $i = 1, \ldots, k$ and $a_k = \lambda_k$. Hence the number of non-vanishing components of the Dynkin label $(\!(a_1, \ldots, a_k)\!)$ coincides with the number of distinct parts of the partition $(\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ with $\ell(\lambda) \leqslant k-1$, that is $b(\lambda) = d(\lambda)$.

Turning now to symmetrized squares, the algebra of Schur functions is such that:

$$\{\lambda\} \times \{\lambda\} = \{\lambda\} \otimes p_1^2 = \{\lambda\} \otimes \{2\} + \{\lambda\} \otimes \{1^2\} \tag{5.5a}$$

$$\{\lambda\} \otimes p_2 = \{\lambda\} \otimes \{2\} - \{\lambda\} \otimes \{1^2\} \tag{5.5b}$$

where p_1 and p_2 are power sum functions (Littlewood 1950, Macdonald 1995). Thus if $\{\lambda\}$ is selfcontragredient

$$\langle \{\lambda\} \otimes \{2\}, \theta \rangle = \tfrac{1}{2}(d(\lambda) + \langle \{\lambda\} \otimes p_2, \theta \rangle) \tag{5.6a}$$

$$\langle \{\lambda\} \otimes \{1^2\}, \theta \rangle = \tfrac{1}{2}(d(\lambda) - \langle \{\lambda\} \otimes p_2, \theta \rangle) \tag{5.6b}$$

where it is now appropriate to make use of the freedom in the choice of Schur functions corresponding to characters of $SU(k+1)$ to take $\theta = \{\theta_\lambda\}$, with the partition θ_λ defined by $\theta_\lambda = (\lambda_1 + 1, \lambda_1^{k-1}, \lambda_1 - 1)$. This ensures that for any selfcontragredient $\{\lambda\}$, for which λ necessarily has weight $w_\lambda = \frac{1}{2}(k+1)\lambda_1$, the Kronecker square $\{\lambda\} \times \{\lambda\} = \{\lambda \cdot \lambda\}$ contains $\{\theta_\lambda\}$ with θ_λ of weight $(k+1)\lambda_1$, without the necessity of modification.

A very efficient method of evaluating $\{\lambda\} \otimes p_r$ has been provided by Littlewood (1951) for any positive integer r. This involves the notions of r-core (or r-residue), r-sign and r-quotient of an arbitrary partition μ. In the case $r = 2$ Littlewood's key theorem implies (Carré and Leclerc 1995, Yang and Wybourne 1995) the following:

Lemma 5.1. Let the 2-core of μ be $\hat{\mu}$, let the 2-sign of μ be $\mathrm{sign}(\mu)$, and let the 2-quotient of μ be $\mathrm{sign}(\mu)\mu^{(0)}\mu^{(1)}$, then

$$\langle\{\lambda\} \otimes p_2, \{\mu\}\rangle = \epsilon_\mu \langle\{\mu^{(0)}\} \cdot \{\mu^{(1)}\}, \{\lambda\}\rangle \tag{5.7}$$

where if $\hat{\mu} \neq 0$ then $\epsilon_\mu = 0$, while if $\hat{\mu} = 0$ then $\epsilon_\mu = \pm 1$ according as the $\mathrm{sign}(\mu) = \pm$.

In the case of interest, namely $\mu = \theta = \theta_\lambda$, we have the following (Yang and Wybourne 1995).

Proposition 5.2. For $\lambda_1 \geqslant 1$ let $\theta_\lambda = (\lambda_1 + 1, \lambda_1^{k-1}, \lambda_1 - 1)$. Then

$$\epsilon_{\theta_\lambda} = \begin{cases} 0 & \text{if } k = 0, 2 (\mathrm{mod}\, 4) \\ -1 & \text{if } k = 3 (\mathrm{mod}\, 4) \\ -(-1)^{\lambda_1} & \text{if } k = 1 (\mathrm{mod}\, 4) \end{cases} \tag{5.8}$$

while for k odd and λ_1 even

$$\{\theta_\lambda^{(0)}\} = \left\{\left(\frac{\lambda_1 + 2}{2}\right)^{(k+1)/2}\right\} \qquad \text{and} \qquad \{\theta_\lambda^{(1)}\} = \left\{\left(\frac{\lambda_1 - 2}{2}\right)^{(k+1)/2}\right\} \tag{5.9a}$$

and for k odd and λ_1 odd

$$\{\theta_\lambda^{(0)}\} = \left\{\left(\frac{\lambda_1 - 1}{2}\right)^{(k+1)/2}\right\} \qquad \text{and} \qquad \{\theta_\lambda^{(1)}\} = \left\{\left(\frac{\lambda_1 + 1}{2}\right)^{(k+1)/2}\right\}. \tag{5.9b}$$

It is particularly noteworthy that the partitions $\theta_\lambda^{(0)}$ and $\theta_\lambda^{(1)}$ are rectangular in that they define rectangular Young diagrams. This ensures that the 2-quotients are easy to write down since for any pair of such partitions $\alpha = (a^p)$ and $\beta = (b^q)$, with $a \geqslant b$ and $p \geqslant q$, we have

$$\{\alpha\} \times \{\beta\} = \{(a^p) \cdot (b^q)\} = \sum_\gamma \{(\alpha + \gamma, \beta/\gamma)\} \tag{5.10}$$

where the summation is over all those partitions γ such that $\gamma_1 \leqslant b$ and $\ell(\gamma) \leqslant q$. Moreover the quotient β/γ consists of a single term δ whose parts are given by $\delta_r = b - \gamma_{q-r+1}$ for $r = 1, \ldots, q$, and the multiplicity of each term $\{\nu\} = \{(\alpha + \gamma, \delta)\}$ in (5.10) is just one.

To evaluate $\{\theta_\lambda^{(0)}\} \cdot \{\theta_\lambda^{(0)}\}$ using (5.10) we choose, for k odd and λ_1 even, $\alpha = (a^p) = \theta_\lambda^{(0)}$ and $\beta = (b^q) = \theta_\lambda^{(1)}$ with $\theta_\lambda^{(0)}$ and $\theta_\lambda^{(1)}$ defined by (5.9a), while for k odd and λ_1 odd we choose $\alpha = (a^p) = \theta_\lambda^{(1)}$ and $\beta = (b^q) = \theta_\lambda^{(0)}$, with $\theta_\lambda^{(0)}$ and $\theta_\lambda^{(1)}$ now defined by (5.9b). In both cases we have $a + b = \lambda_1$ and $p = q = \frac{1}{2}(k + 1)$. If we set $\nu = (\alpha + \gamma, \delta)$ with $\delta = \beta/\gamma$, then $\nu_i + \nu_{k-i+2} = \nu_i + \nu_{2p-i+1} = a + \gamma_i + \delta_{p-i+1} = a + \gamma_i + b - \gamma_i = a + b = \lambda_1$ for $i = 1, \ldots, (k + 1)/2$. This implies that the corresponding irreducible representation $\{\nu\}$ of $SU(k + 1)$ is necessarily selfcontragredient. Moreover its multiplicity in $\{\theta_\lambda^{(0)}\} \cdot \{\theta_\lambda^{(1)}\}$ is one.

In fact this product contains almost all selfcontragredient irreducible representations $\{\lambda\}$ with $\ell(\lambda) \leqslant k$ for k odd and fixed λ_1. This can be seen by noting that for k odd the conditions that ensure $\{\lambda\}$ is a selfcontragredient irreducible representation of $SU(k + 1)$ given in table 4 imply that λ is of the form $(\alpha + \gamma, \delta)$ with $\delta = \beta/\gamma$, where $\alpha = (a^p)$ and $\beta = (b^q)$ with $p = q = \frac{1}{2}(k + 1)$, $a + b = \lambda_1$ and $a = [(\lambda_1 + 1)/2]$. If λ_1 is odd this set of all selfcontragredient irreducible representations $\{\lambda\}$ coincides with the set of all $\{\nu\}$ obtained previously from (5.10) with parameters determined by (5.9b) so that $a = (\lambda_1 + 1)/2$. However, if λ_1 is even the set of all $\{\nu\}$ obtained previously from (5.10) with parameters determined by (5.9a) is such that $a = (\lambda_1 + 2)/2$ rather than $\lambda_1/2$ as required to exhaust all possible selfcontragredient irreducible representations $\{\lambda\}$ with λ_1 even. The only selfcontragredient irreducible representations $\{\lambda\}$ missing from $\{\theta_\lambda^{(0)}\} \cdot \{\theta_\lambda^{(1)}\}$ are therefore those for which λ_1 is even and $\lambda_{(k+1)/2} = \frac{1}{2}\lambda_1$.

These remarks taken in conjunction with lemma 5.1 then imply:

Proposition 5.3. For $\lambda_1 \geqslant 1$ let $\theta_\lambda = (\lambda_1 + 1, \lambda_1^{k-1}, \lambda_1 - 1)$. Then the irreducible representations $\{\lambda\}$ and $\{\theta_\lambda\}$ of $SU(k+1)$ are such that $\langle\{\lambda\} \otimes p_2, \{\theta_\lambda\}\rangle$ is zero unless $\{\lambda\}$ is selfcontragredient, k is odd, and $\lambda_{(k+1)/2} \neq \frac{1}{2}\lambda_1$. If these three conditions are satisfied then

$$\langle\{\lambda\} \otimes p_2, \{\theta_\lambda\}\rangle = \begin{cases} -1 & \text{if } k = 1(\text{mod}\,4),\ \lambda_1 = 0(\text{mod}\,2) \\ & \text{or if } k = 3(\text{mod}\,4) \\ 1 & \text{if } k = 1(\text{mod}\,4),\ \lambda_1 = 1(\text{mod}\,2). \end{cases} \tag{5.11}$$

This in turn allows us to conclude the validity of the following result which in the case of $SU(k+1)$ adds very significantly to proposition 4.2 by specifying precisely how the $b(\lambda)$ copies of the adjoint irreducible representation θ are distributed between the symmetric and antisymmetric parts of the Kronecker square of a selfcontragredient irreducible representation $\{\lambda\}$. It should be noted that $b(\{\lambda\}) = d(\lambda)$, the number of distinct parts of the partition λ.

Proposition 5.4. Let $\{\lambda\}$ and θ be an arbitrary finite-dimensional selfcontragredient irreducible representation and the adjoint irreducible representation, respectively, of $SU(k+1)$. Then

$$\langle\{\lambda\} \otimes \{2\}, \theta\rangle = \begin{cases} \frac{1}{2}d(\lambda) & \text{if } d(\lambda) \text{ is even} \\ \frac{1}{2}(d(\lambda) - 1) & \text{if } d(\lambda) \text{ is odd and } \{\lambda\} \text{ is orthogonal} \\ \frac{1}{2}(d(\lambda) + 1) & \text{if } d(\lambda) \text{ is odd and } \{\lambda\} \text{ is symplectic} \end{cases}$$

$$\langle\{\lambda\} \otimes \{1^2\}, \theta\rangle = \begin{cases} \frac{1}{2}d(\lambda) & \text{if } d(\lambda) \text{ is even} \\ \frac{1}{2}(d(\lambda) + 1) & \text{if } d(\lambda) \text{ is odd and } \{\lambda\} \text{ is orthogonal} \\ \frac{1}{2}(d(\lambda) - 1) & \text{if } d(\lambda) \text{ is odd and } \{\lambda\} \text{ is symplectic.} \end{cases}$$

Proof. If $\{\lambda\}$ is selfcontragredient but k is even then from table 5 it is clear that $d(\lambda)$ is even. In addition if $\{\lambda\}$ is selfcontragredient and k is odd but $\lambda_{(k+1)/2} = \frac{1}{2}\lambda_1$ with λ_1 even, then from the conditions of table 4 we must also have $\lambda_{(k+3)/2} = \frac{1}{2}\lambda_1$ so that $a_{(k+1)/2} = 0$. Remembering that $\{\lambda\}$ is selfcontragredient, this implies once more that $d(\lambda)$ is even. It therefore follows from proposition 5.3 that $\langle\{\lambda\} \otimes p_2, \{\theta_\lambda\}\rangle$ is non-zero if and only if $d(\lambda)$ is odd. Moreover the conditions appearing in (5.11) are precisely those appropriate to distinguish between orthogonal and symplectic irreducible representation $\{\lambda\}$ as spelled out in table 5. Proposition 5.4 then follows from the application of proposition 5.3 to (5.6*a*) and (5.6*b*).

A related approach to the derivation of proposition 5.4 has been given by Yang and Wybourne (Yang and Wybourne 1995), who did not, however, make the connection with the evenness or oddness of $d(\lambda)$ and the orthogonal or symplectic nature of $\{\lambda\}$. While Carre and Leclerc (1995) have derived a combinatorial algorithm for the complete resolution of the symmetric and antisymmetric squares of any irreducible representation λ, this algorithm does not appear to provide any way of arriving at proposition 5.4, or indeed its precursor Proposition 5.3, which is as simple as the use of proposition 5.2 and the exploitation of (5.10). $\square$

6. Symmetrized Kronecker squares for $SO(2k+1)$, $Sp(2k)$ and $SO(2k)$

For the classical compact simple Lie groups other than $SU(k+1)$ it is convenient to adopt a completely different approach. This is motivated by the fact that in numerous examples it has been found that the occurrence of the ajoint irreducible representation θ in the symmetric and antisymmetric squares of a given irreducible representation λ has been confined to one or other of these two parts, but not both. We concentrate therefore on establishing that the multiplicity of θ in one or other of $\lambda \otimes \{2\}$ and $\lambda \otimes \{1^2\}$ is zero for an arbitrary selfcontragredient irreducible representation λ.

For any such irreducible representation λ the starting point is somewhat surprisingly the consideration of the Kronecker product $\lambda \times \omega$, where for each of the groups $SO(2k+1)$, $Sp(2k)$ and $SO(2k)$ we take $\omega = \omega_1$, the appropriate defining irreducible representation given in terms of natural labels by [1], $\langle 1 \rangle$ and [1], respectively. The relevant products with λ may be evaluated either through the use of Weyl's character formula for $ch\, V^\lambda$ and a knowledge of the weights of ω, or by means of Schur function techniques (King *et al* 1981, Black *et al* 1983). We obtain using the latter the results of table 5.

Table 5. The Schur function decomposition of Kronecker products of the form $\lambda \times \omega$ for $SO(2k+1)$, $Sp(2k)$ and $SO(2k)$.

$\mathcal{G}$	$\lambda \times \omega$	Constraints
$SO(2k+1)$	$[\lambda] \times [1] = [\lambda \cdot 1] + [\lambda/1]$	$\ell(\lambda) \leqslant k$
	with $[\lambda, 1] = [\lambda]$ if $\ell(\lambda) = k$	
	$[\Delta; \lambda] \times [1] = [\Delta; \lambda \cdot 1] + [\Delta; \lambda] + [\Delta; \lambda/1]$	$\ell(\lambda) \leqslant k$
	with $[\Delta; \lambda, 1] = 0$ if $\ell(\lambda) = k$	
$Sp(2k)$	$\langle\lambda\rangle \times \langle 1\rangle = \langle\lambda \cdot 1\rangle + \langle\lambda/1\rangle$	$\ell(\lambda) \leqslant k$
	with $\langle\lambda, 1\rangle = 0$ if $\ell(\lambda) = k$	
$SO(2k)$	$[\lambda] \times [1] = [\lambda \cdot 1] + [\lambda/1]$	$\ell(\lambda) < k$
	with $[\lambda, 1] = [\lambda, 1]_+ + [\lambda, 1]_-$ if $\ell(\lambda) = k - 1$	
	$[\lambda]_\pm \times [1] = [\lambda \cdot 1]_\pm + [\lambda/1]_\pm$	$\ell(\lambda) = k$
	with $[\lambda, 1]_\pm = 0$ and $[\mu]_\pm = [\mu]$ if $\ell(\mu) < k$	
	$[\Delta; \lambda]_\pm \times [1] = [\Delta; \lambda \cdot 1]_\pm + [\Delta; \lambda]_\mp + [\Delta; \lambda/1]_\pm$	$\ell(\lambda) \leqslant k$
	with $[\Delta; \lambda, 1]_\pm = [\Delta; \lambda]_\mp = 0$ if $\ell(\lambda) = k$	

The single most notable thing about these results is that all the products are multiplicity free as a consequence of lemma 3.1 and the fact that all the terms in $\lambda \cdot 1$, λ and $\lambda/1$ are specified by partitions of weight $w_\lambda + 1$, w_λ and $w_\lambda - 1$, respectively. This conclusion remains valid even in those special cases for which it is necessary to invoke the modification rules included in table 6.

With the exception of the case $SO(2k)$ with k odd all the irreducible representations appearing as constituents of each product $\lambda \times \omega$ are selfcontragredient. In this exceptional case we can of course restrict ourselves to the product $[\lambda] \times [1]$ with $\ell(\lambda) < k$ since both $[\lambda]_\pm$ and $[\Delta; \lambda]_\pm$ are not selfcontragredient, as made clear in table 3. However, if $\ell(\lambda) = k - 1$ then the product $[\lambda] \times [1]$ contains the pair of mutually contragredient irreducible representations $[\lambda, 1]_+$ and $[\lambda, 1]_-$. It is therefore necessary to exclude $SO(2k)$ with k odd from the following lemma.

Lemma 6.1. In the case of $SO(2k+1)$, $Sp(2k)$ and $SO(2k)$ with k even:

(i) the Kronecker product $\lambda \times \omega$ of an arbitrary irreducible representation λ and the defining irreducible representation ω decomposes into a direct sum $\mu + \nu + \cdots +$ of mutually distinct, selfcontragredient irreducible representations;

(ii) these irreducible representations μ, ν, ... are all orthogonal if λ and ω are either both orthogonal or both symplectic, and are all symplectic if λ is orthogonal and ω is symplectic or vice versa;

(iii) the multiplicity of occurrence of the identity irreducible representation η in the symmetric and antisymmetric square of $\lambda \times \omega$ is such that:

$$\langle(\lambda \times \omega) \otimes \{1^2\}, \eta\rangle = 0 \qquad \text{if} \begin{cases} \lambda \text{ and } \omega \text{ are both orthogonal, or} & (6.1b) \\ \lambda \text{ and } \omega \text{ are both symplectic} & (6.1b) \end{cases}$$

$$\langle(\lambda \times \omega) \otimes \{2\}, \eta\rangle = 0 \qquad \text{if} \begin{cases} \lambda \text{ is orthogonal and } \omega \text{ is symplectic, or} & (6.1c) \\ \lambda \text{ is symplectic and } \omega \text{ is orthogonal.} & (6.1d) \end{cases}$$

Proof. Part (i) follows from our previous remarks, while part (ii) is an immediate consequence of the work of Mal'cev (1962), see also Adams (1969), applied to products in which the irreducible constituents are selfcontragredient and distinct. In part (iii) it then follows that if, for example, λ and ω are both orthogonal, then from (3.7b)

$$\begin{aligned}\langle(\lambda \times \omega) \otimes \{1^2\}, \eta\rangle &= \langle(\mu + \nu + \cdots) \otimes \{1^2\}, \eta\rangle \\ &= \langle\mu \otimes \{1^2\}, \eta\rangle + \langle\nu \otimes \{1^2\}, \eta\rangle + \cdots + \langle\mu \times \nu, \eta\rangle + \cdots. \qquad (6.2)\end{aligned}$$

However, since μ, ν, $\ldots$ are all distinct, selfcontragredient, orthogonal irreducible representations, then $\langle\mu \otimes \{1^2\}, \eta\rangle = \langle\nu \otimes \{1^2\}, \eta\rangle = \ cdots = 0$ from propositions 5.1 and 5.2, and $\langle\mu \times \nu, \eta\rangle = \cdots = 0$ from (3.9). Thus all terms contributing to (6.2) are identically zero and the result (6.1a) follows. The results (6.1b), (6.1c) and (6.1d) can all be proved in the same way. □

This leads inexorably to the following result:

Proposition 6.2. If λ is an arbitrary irreducible representation of $SO(2k+1)$ or $SO(2k)$ with k even, then

$$\begin{aligned}&\langle(\lambda \otimes \{2\}), [1^2]\rangle = \langle(\lambda \otimes \{1^2\}), [2]\rangle = 0 \qquad \text{if } \lambda \text{ is orthogonal} \qquad (6.3a) \\ &\langle(\lambda \otimes \{1^2\}), [1^2]\rangle = \langle(\lambda \otimes \{2\}), [2]\rangle = 0 \qquad \text{if } \lambda \text{ is symplectic.} \qquad (6.3b)\end{aligned}$$

If λ is an arbitrary irreducible representation of $Sp(2k)$, then

$$\begin{aligned}&\langle(\lambda \otimes \{2\}), \langle 2\rangle\rangle = \langle(\lambda \otimes \{1^2\}), \langle 1^2\rangle\rangle = 0 \qquad \text{if } \lambda \text{ is orthogonal} \qquad (6.4a) \\ &\langle(\lambda \otimes \{1^2\}), \langle 2\rangle\rangle = \langle(\lambda \otimes \{2\}), \langle 1^2\rangle\rangle = 0 \qquad \text{if } \lambda \text{ is symplectic.} \qquad (6.4b)\end{aligned}$$

Proof. In the case of $SO(2k+1)$ or $SO(2k)$ with k even the defining irreducible representation $\omega = [1]$ is orthogonal. Let the irreducible representation λ also be orthogonal in accordance with case (6.1a). From (3.7b) we have

$$\begin{aligned}\langle(\lambda \times [1]) \otimes \{1^2\}, \eta\rangle &= \langle(\lambda \otimes \{1^2\}) \times ([1] \otimes \{2\}), \eta\rangle + \langle(\lambda \otimes \{2\}) \times ([1] \otimes \{1^2\}), \eta\rangle \\ &= \langle(\lambda \otimes \{1^2\}) \times ([0] + [2]), \eta\rangle + \langle(\lambda \otimes \{2\}) \times ([1^2]), \eta\rangle \\ &= \langle\lambda \otimes \{1^2\}, [0]\rangle + \langle\lambda \otimes \{1^2\}, [2]\rangle + \langle\lambda \otimes \{2\}, [1^2]\rangle \qquad (6.5)\end{aligned}$$

where use has been made of the decompositions $[1]\otimes\{2\} = [0]+[2]$ and $[1]\otimes\{1^2\} = [1^2]$. As we are assuming that the irreducible representation λ is orthogonal then $\langle\lambda \otimes \{1^2\}, [0]\rangle = 0$. However, the left-hand side of (6.5) vanishes identically by (6.1a) so that all terms on the right must be zero, giving (6.3a) as required. The result (6.3b) may be derived in the same way, making use this time of (3.7a). Similarly if λ is an irreducible representation of $Sp(2k)$ and $\omega = \langle 1\rangle$ both (6.4a) and (6.4b) may be derived in the same way using (3.7a) or (3.7b) as appropriate, along with $\langle 1\rangle \otimes \{2\} = \langle 2\rangle$ and $\langle 1\rangle \otimes \{1^2\} = \langle 0\rangle + \langle 1^2\rangle$. □

Recalling the result $\langle\lambda\times\lambda,\theta\rangle = b(\lambda)$ from proposition 4.2 and the relationship between $b(\!(a_1,\ldots,a_k)\!)$ and $d(\lambda)$ implied by the connection between Dynkin and natural labels, we can summarize our results as follows.

Proposition 6.3. The multiplicities of the adjoint irreducible representation $\theta = [1^2]$ in the symmetric and antisymmetric squares of the irreducible representations $[\lambda]$ and $[\Delta;\lambda]$ of $SO(2k+1)$ are given by:

$$\langle[\lambda]\otimes\{2\},[1^2]\rangle = 0 \qquad \text{and} \qquad \langle[\lambda]\otimes\{1^2\},[1^2]\rangle = d(\lambda) \tag{6.6a}$$

and

$$\langle[\Delta;\lambda]\otimes\{2\},[1^2]\rangle = \begin{cases} 0 & \text{if } [\Delta;\lambda] \text{ is orthogonal} \\ d(\lambda)+1 & \text{if } [\Delta;\lambda] \text{ is symplectic and } \ell(\lambda)<k \\ d(\lambda) & \text{if } [\Delta;\lambda] \text{ is symplectic and } \ell(\lambda)=k \end{cases} \tag{6.6b}$$

$$\langle[\Delta;\lambda]\otimes\{1^2\},[1^2]\rangle = \begin{cases} d(\lambda)+1 & \text{if } [\Delta;\lambda] \text{ is orthogonal and } \ell(\lambda)<k \\ d(\lambda) & \text{if } [\Delta;\lambda] \text{ is orthogonal and } \ell(\lambda)=k \\ 0 & \text{if } [\Delta;\lambda] \text{ is symplectic.} \end{cases} \tag{6.6c}$$

Proposition 6.4. The multiplicities of the adjoint irreducible representation $\theta = \langle 2\rangle$ in the symmetric and antisymmetric squares of the irreducible representations $\langle\lambda\rangle$ of $Sp(2k)$ are given by:

$$\langle\langle\lambda\rangle\otimes\{2\},\langle 2\rangle\rangle = \begin{cases} 0 & \text{if } \langle\lambda\rangle \text{ is orthogonal} \\ d(\lambda) & \text{if } \langle\lambda\rangle \text{ is symplectic} \end{cases} \tag{6.7a}$$

$$\langle\langle\lambda\rangle\otimes\{1^2\},\langle 2\rangle\rangle = \begin{cases} d(\lambda) & \text{if } \langle\lambda\rangle \text{ is orthogonal} \\ 0 & \text{if } \langle\lambda\rangle \text{ is symplectic.} \end{cases} \tag{6.7b}$$

Proposition 6.5. The multiplicities of the adjoint irreducible representation $\theta = [1^2]$ in the symmetric and antisymmetric squares of the irreducible representations $[\lambda]$, $[\lambda]_\pm$ and $[\Delta;\lambda]_\pm$ of $SO(2k)$ with k even are given by:

$$\langle[\lambda]\otimes\{2\},[1^2]\rangle = 0 \qquad \text{and} \qquad \langle[\lambda]\otimes\{1^2\},[1^2]\rangle = d(\lambda) \qquad \text{for } \ell(\lambda)<k \tag{6.8a}$$

$$\langle[\lambda]_\pm\otimes\{2\},[1^2]\rangle = 0 \qquad \text{and} \qquad \langle[\lambda]_\pm\otimes\{1^2\},[1^2]\rangle = d(\lambda) \qquad \text{for } \ell(\lambda)=k \tag{6.8b}$$

and

$$\langle[\Delta;\lambda]_\pm\otimes\{2\},[1^2]\rangle = \begin{cases} 0 & \text{if } [\Delta;\lambda]_\pm \text{ is orthogonal} \\ d(\lambda)+1 & \text{if } [\Delta;\lambda]_\pm \text{ is symplectic and } \ell(\lambda)<k \\ d(\lambda) & \text{if } [\Delta;\lambda]_\pm \text{ is symplectic and } \ell(\lambda)=k \end{cases} \tag{6.8c}$$

$$\langle[\Delta;\lambda]_\pm\otimes\{1^2\},[1^2]\rangle = \begin{cases} d(\lambda)+1 & \text{if } [\Delta;\lambda]_\pm \text{ is orthogonal and } \ell(\lambda)<k \\ d(\lambda) & \text{if } [\Delta;\lambda]_\pm \text{ is orthogonal and } \ell(\lambda)=k \\ 0 & \text{if } [\Delta;\lambda]_\pm \text{ is symplectic.} \end{cases} \tag{6.8d}$$

Returning to the troublesome case of $SO(2k)$ with k odd, the only selfcontragredient irreducible representations are those irreducible representations $[\lambda]$ for which $\ell(\lambda)<k$. In the case $\ell(\lambda)<k-1$ everything goes through as before with all terms contributing to (6.2) vanishing, so that (6.1a) is still valid and implies (6.3a). Difficulties are encountered only in the case $\ell(\lambda)=k-1$. In this case, over and above well-behaved distinct, selfcontragredient, orthogonal irreducible representations $[\mu]$, $[\nu]$, ... with $\ell(\mu)$, $\ell(\nu)$, ... all less than k in

(6.2), there now appear the two terms $[\lambda, 1]_+$ and $[\lambda, 1]_-$. At first sight the presence of these irreducible representations seems harmless enough since they are not selfcontragredient and $\langle [\lambda, 1]_+ \otimes \{1^2\}, \eta\rangle = \langle [\lambda, 1]_- \otimes \{1^2\}, \eta\rangle = 0$. On the other hand they are contragredients of one another, so that from (3.9) we have $\langle([\lambda, 1]_+ \times [\lambda, 1]_-), \eta\rangle = 1$. Hence for $SO(2k)$ with k odd, (6.1*a*) must be replaced by:

$$\langle([\lambda] \times [1]) \otimes \{1^2\}, \eta\rangle = \begin{cases} 0 & \text{if } \ell(\lambda) < k-1 \\ 1 & \text{if } \ell(\lambda) = k-1. \end{cases} \tag{6.9}$$

By means of (6.5) we can therefore only conclude in place of (6.3*a*) that

$$\langle([\lambda] \otimes \{2\}), [1^2]\rangle = \langle([\lambda] \otimes \{1^2\}), [2]\rangle = 0 \quad \text{if } \ell(\lambda) < k-1 \tag{6.10a}$$

$$\langle([\lambda] \otimes \{2\}), [1^2]\rangle + \langle([\lambda] \otimes \{1^2\}), [2]\rangle = 1 \quad \text{if } \ell(\lambda) = k-1. \tag{6.10b}$$

The last equation has two solutions:

$$\text{(A)} \qquad \langle([\lambda] \otimes \{2\}), [1^2]\rangle = 1 \qquad \text{and} \qquad \langle([\lambda] \otimes \{1^2\}), [2]\rangle = 0 \tag{6.11a}$$

$$\text{(B)} \qquad \langle([\lambda] \otimes \{2\}), [1^2]\rangle = 0 \qquad \text{and} \qquad \langle([\lambda] \otimes \{1^2\}), [2]\rangle = 1. \tag{6.11b}$$

It is not difficult by exploiting the isomorphism between $SO(6)$ and $SU(4)$ to show that for $k = 3$ the solution (A) applies to all $[\lambda]$ such that $\ell(\lambda) = 2$. Similarly for all $SO(2k)$ with k odd solution (A) also applies in the case $[\lambda] = [1^{k-1}]$. The problem may be unequivocally resolved by considering the case of the full orthogonal group $O(2k)$ with k odd and then restricting to its subgroup $SO(2k)$. The key result takes the form:

Lemma 6.6. The multiplicities of the adjoint irreducible representation $\theta = [1^2]$ and its associate $\theta^* = [1^2]^* = [1^{2k-2}]$ in the symmetric and antisymmetric squares of the irreducible representations $[\lambda]$ of $O(2k)$ with k odd and $\ell(\lambda) = k-1$ are given by:

$$\langle[\lambda] \otimes \{2\}, [1^2]\rangle = 0 \qquad \text{and} \qquad \langle[\lambda] \otimes \{1^2\}, [1^2]\rangle = d(\lambda) \tag{6.12a}$$

$$\langle[\lambda] \otimes \{2\}, [1^2]^*\rangle = 1 \qquad \text{and} \qquad \langle[\lambda] \otimes \{1^2\}, [1^2]^*\rangle = 0. \tag{6.12b}$$

Proof. First it should be noted that each irreducible representation $[\lambda]$ of $O(2k)$ with $\ell(\lambda) = k-1$ is orthogonal and possesses an inequivalent associate irreducible representation $[\lambda]^* = [\lambda, 1^2]$ (King *et al* 1981). Moreover,

$$[\lambda] \times [1^2] = [\lambda \cdot 1^2] + [\lambda/1 \cdot 1] + [\lambda/1^2]. \tag{6.13}$$

Recalling lemma 3.2 and noting that $[\lambda \cdot 1^2]$ contains $[\lambda, 1^2] = [\lambda]^*$ with multiplicity one, it follows that

$$\langle[\lambda] \times [1^2], [\lambda]\rangle = d(\lambda) \qquad \text{and} \qquad \langle[\lambda] \times [1^2], [\lambda]^*\rangle = 1. \tag{6.14}$$

Since both $[\lambda]$ and $[1^2]$ are selfcontragredient lemma 3.5 then implies that

$$\langle[\lambda] \times [\lambda], [1^2]\rangle = d(\lambda) \qquad \text{and} \qquad \langle[\lambda] \times [\lambda], [1^2]^*\rangle = 1 \tag{6.15}$$

where use has also been made of the fact that $[\mu]^* = [0]^* \times [\mu]$ for all $[\mu]$, where $[0]^* = [1^{2k}]$ is the irreducible representation of $O(2k)$ which maps each group element to its determinant, ± 1 (King *et al* 1981). It should be noted that $[1^2]^* = [1^{2k-2}]$.

The second of the two results in (6.15) can be derived more directly by noting that

$$[\lambda] \times [\lambda] = \sum_{\xi} [\lambda/\xi \cdot \lambda/\xi]. \tag{6.16}$$

Recalling that by hypothesis $\ell(\lambda) = k-1$, it is clear that even taking modification rules (King *et al* 1981) into account, the only way that a term $[1^2]^* = [1^{2k-2}]$ can arise on the right-hand side of (6.16) is if λ/ξ contains 1^{k-1} for some ξ. Since $\ell(\lambda) = k-1$ such a ξ exists and is unique. In fact $\xi = \lambda/1^{k-1}$ and $\lambda/\xi = 1^{k-1}$. Hence as claimed $\langle[\lambda] \times [\lambda], [1^2]^*\rangle = 1$.

Turning to symmetrized products, it should be noted first that the difficulties referred to above which arise in the case of $SO(2k)$ with k odd, do not arise in the case of $O(2k)$ with k odd. In particular for our orthogonal irreducible representation $\lambda = [\lambda]$ of $O(2k)$ with $\ell(\lambda) = k-1$ and $\omega = [1]$ (6.1*a*) is valid since now in (6.2) the term $[\lambda, 1]$ is irreducible and orthogonal. Thanks to (6.5) this in turn implies the validity of (6.3*a*). Combining this with the first part of (6.15) then gives (6.12*a*). Finally, Littlewood's theorem III (Littlewood 1958) implies that

$$\langle[\lambda] \otimes \{2\}, [1^2]^*\rangle = \langle\{1^{k-1}\} \otimes \{2\}, \{1^{2k-2}\}\rangle \tag{6.17a}$$

$$\langle[\lambda] \otimes \{1^2\}, [1^2]^*\rangle = \langle\{1^{k-1}\} \otimes \{1^2\}, \{1^{2k-2}\}\rangle. \tag{6.17b}$$

However, if k is odd $\langle\{1^{k-1}\} \otimes \{2\}, \{1^{2k-2}\}\rangle = 1$, and $\langle\{1^{k-1}\} \otimes \{1^2\}, \{1^{2k-2}\}\rangle = 0$ (King *et al* 1981). Combining (6.17) with the second part of (6.15) then gives (6.12*b*). □

The validity of lemma 6.6 then allows us to complete the analysis of $SO(2k)$ with k odd by means of the following:

Proposition 6.7. The multiplicities of the adjoint irreducible representation $\theta = [1^2]$ in the symmetric and antisymmetric squares of the irreducible representations $[\lambda]$ of $SO(2k)$ with k odd and $\ell(\lambda) < k$ are given by:

$$\langle[\lambda] \otimes \{2\}, [1^2]\rangle = \begin{cases} 0 & \text{if } \ell(\lambda) < k-1 \\ 1 & \text{if } \ell(\lambda) = k-1 \end{cases} \tag{6.18a}$$

$$\langle[\lambda] \otimes \{1^2\}, [1^2]\rangle = d(\lambda). \tag{6.18b}$$

Proof. As we have already indicated, the case $\ell(\lambda) < k-1$ gives no problem, and the required result follows from the use of (6.2) to give (6.1*a*), the subsequent use of (6.5) to give (6.3*a*) and the observation that $\langle[\lambda] \times [\lambda], [1^2]\rangle = b(\lambda) = d(\lambda)$. The case $\ell(\lambda) = k-1$ follows directly from lemma 6.6 and the observation that under the restriction from $O(2k)$ to $SO(2k)$ we have $[\lambda] \to [\lambda]$ for $\ell(\lambda) = k-1$ and $[1^2]^* \to [1^2]$. □

7. Symmetrized Kronecker squares for the exceptional groups

The techniques of section 6 are also appropriate for use with some of the exceptional groups. The trick is to find some irreducible representation ω which is selfcontragredient, so that its square contains the adjoint θ, and whose weights are multiplicity free, so that the decomposition of $\lambda \times \omega$ is likely to be multiplicity free for all λ. Even this may not be enough as we have seen in the case of $SO(2k)$ with k odd for which the confounding factor was the occurrence of mutually contragredient pairs of distinct irreducible representations in $\lambda \times \omega$.

However for G_2, whose adjoint irreducible representation is (21), it is helpful to consider the product of an arbitrary irreducible representation (λ) with the defining irreducible representation $\omega = \omega_1 = (1)$. This irreducible representation (1) is orthogonal and its weights all have multiplicity one. In addition all the irreducible representations of G_2 are selfcontragredient. The product $\lambda \times \omega$ takes the form (King 1981):

$$(\lambda) \times (1) = \begin{cases} (\lambda \cdot 1) + (\lambda \cdot 1^2) + (\lambda) & \text{if } \lambda_1 \geqslant 2\lambda_2 \\ (\lambda \cdot 1) + (\lambda \cdot 1^2) & \text{if } \lambda_1 = 2\lambda_2 \end{cases} \tag{7.1}$$

where the Schur function products are to be evaluated as products of characters of $SU(3)$ but with any term $(\mu) = (\mu_1, \mu_2)$ discarded if $\mu_1 < 2\mu_2$. It can be seen that in all cases $(\lambda) \times (1)$ decomposes into a sum of distinct, selfcontragredient, orthogonal irreducible representations. It follows, by the same argument that was applied to (6.2), that:

$$\langle ((\lambda) \times (1)) \otimes \{1^2\}, \eta \rangle = 0. \tag{7.2}$$

As in (6.5) we now obtain

$$\begin{aligned} &\langle ((\lambda) \times (1)) \otimes \{1^2\}, \eta \rangle \\ &\quad = \langle (\lambda) \otimes \{1^2\}) \times (1) \otimes \{2\}), \eta \rangle + \langle (\lambda) \otimes \{2\}) \times (1) \otimes \{1^2\}), \eta \rangle \\ &\quad = \langle ((\lambda) \otimes \{1^2\}) \times (0) + (2)), \eta \rangle + \langle ((\lambda) \otimes \{2\}) \times (1) + (21)), \eta \rangle \\ &\quad = \langle (\lambda) \otimes \{1^2\}, (2) \rangle + \langle (\lambda) \otimes \{1^2\}, (1) \rangle + \langle (\lambda) \otimes \{2\}, (21) \rangle \end{aligned} \tag{7.3}$$

where use has been made of the decompositions $(1) \otimes \{2\} = (0) + (2)$ and $(1) \otimes \{1^2\} = (1)+(21)$, and the fact that (λ) is orthogonal. It follows that all three terms on the right-hand side of (7.3) must be zero. In particular we have

$$\langle (\lambda) \otimes \{2\}, (21) \rangle = 0. \tag{7.4}$$

Combining this with the results of section 4 and expressing $b((a_1, a_2))$ in terms of λ_1 and λ_2 we arrive at:

Proposition 7.1. The multiplicities of the adjoint irreducible representation $\theta = (21)$ in the symmetric and antisymmetric squares of the irreducible representation (λ) of G_2 are given by:

$$\langle (\lambda) \otimes \{2\}, (21) \rangle = 0 \tag{7.5a}$$

$$\langle (\lambda) \otimes \{1^2\}, (21) \rangle = \begin{cases} 2 & \text{if } \lambda_1 > 2\lambda_2 > 0 \\ 1 & \text{if } \lambda_1 = 2\lambda_2 > 0 \text{ or } \lambda_1 > 2\lambda_2 = 0. \end{cases} \tag{7.5b}$$

Proceeding in exactly the same way for E_7 but now taking $\omega = \omega_7 = (1^6)$ we have (King 1981):

$$(\lambda) \times (1^6) = (\lambda \cdot 1^2) + (\lambda \cdot 1^6) \tag{7.6}$$

where Schur function products are to be evaluated as products of characters of $SU(8)$ but with any term $(\mu) = (\mu_1, \dots, \mu_7)$ discarded if $\mu_1 < \mu_2 + \mu_3 + \mu_4 + \mu_5 - \mu_6 - \mu_7$. Once again all terms in the decomposition (7.6) are distinct, selfcontragredient irreducible representations which are either all orthogonal or all symplectic according as $w_\lambda = 0 (\mathrm{mod}\, 4)$ or $2 (\mathrm{mod}\, 4)$, respectively. Since $\omega = (1^6)$ is symplectic, it follows as in (6.1b) and (6.1c) that

$$\langle ((\lambda) \times (1^6)) \otimes \{1^2\}, \eta \rangle = 0 \qquad \text{if } (\lambda) \text{ is symplectic} \tag{7.7a}$$

$$\langle ((\lambda) \times (1^6)) \otimes \{2\}, \eta \rangle = 0 \qquad \text{if } (\lambda) \text{ is orthogonal.} \tag{7.7b}$$

Using (3.8), the left-hand sides of these two equations may then be expanded as in (6.5) or (7.3), with the symmetric and antisymmetric squares of ω given by (Wybourne and Bowick 1977) $(1^6) \otimes \{2\} = (21^6) + (2^6)$ and $(1^6) \otimes \{1^2\} = (0) + (2^5 1^2)$. This leads to the conclusion that:

$$\langle (\lambda) \otimes \{1^2\}, (21^6) \rangle = 0 \qquad \text{if } (\lambda) \text{ is symplectic} \tag{7.8a}$$

$$\langle (\lambda) \otimes \{2\}, (21^6) \rangle \ = 0 \qquad \text{if } (\lambda) \text{ is orthogonal.} \tag{7.8b}$$

By making use of (3.6) and of proposition 4.2, as applied to E_7, we can infer the following:

Proposition 7.2. The multiplicities of the adjoint irreducible representation $\theta = (21^6)$ in the symmetric and antisymmetric squares of the irreducible representations (λ) of E_7 are given by:

$$\langle(\lambda)\rangle \otimes \{2\}, (21^6)\rangle = \begin{cases} 0 & \text{if } (\lambda) \text{ is orthogonal} \\ b((\lambda)) & \text{if } (\lambda) \text{ is symplectic} \end{cases} \tag{7.9a}$$

$$\langle(\lambda) \otimes \{1^2\}, (21^6)\rangle = \begin{cases} b((\lambda)) & \text{if } (\lambda) \text{ is orthogonal} \\ 0 & \text{if } (\lambda) \text{ is symplectic.} \end{cases} \tag{7.9b}$$

This completes the happy part of the story regarding the exceptional groups. In the case of both F_4 and E_8 it is not possible to find any ω such that for all irreducible representations λ the decomposition of the product $\lambda \times \omega$ is multiplicity free. The best that can be done is to take $\omega = \omega_1 = (1)$ in F_4 and $\omega = \omega_1 = (21^7)$ in E_8. The fact that these irreducible representations have zero weights (0) the multiplicities of which are two and eight, respectively, ensures that in almost all cases

$$\langle(\lambda \times \omega_1) \otimes \{1^2\}, \eta\rangle \neq 0 \tag{7.10}$$

even though all irreducible representations of F_4 and E_8 are orthogonal. This is analogous to the appearance of a non-vanishing term in (6.9) for $SO(2k)$ with k odd. It thwarts our attempt to use (3.8) and the symmetrized Kronecker squares of (1) to separate unambiguously the multiplicities of θ in $\lambda \times \lambda$ into contributions to $\lambda \otimes \{2\}$ and $\lambda \otimes \{1^2\}$. Nonetheless, on the basis of our accumulated data, we are tempted to make the following conjecture.

Conjecture 7.3. The multiplicities of the adjoint irreducible representation θ in the symmetric and antisymmetric squares of the irreducible representations λ of F_4 and E_8 are given by:

$$\langle\lambda \otimes \{2\}, \theta\rangle = 0 \tag{7.11a}$$

$$\langle\lambda \otimes \{1^2\}, \theta\rangle = b(\lambda). \tag{7.11b}$$

The case of E_6 appears to be intractable for a combination of reasons. Firstly it does not possess a selfcontragredient irreducible representation whose weights are multiplicity free. Indeed its simplest selfcontragredient irreducible representation is the adjoint, whose zero weight has multiplicity given by the rank 6. This implies that the techniques used for G_2 for example will not lead to a unique resolution of the multiplicity problem. In this sense it is analogous to F_4 and E_8. However, it is worse since the products $\lambda \times \omega$ contain irreducible representations which are not selfcontragredient even when both λ and ω are orthogonal. In fact E_6 is more closely related to $SU(k+1)$ which we have seen required rather special treatment. In this case we are tempted, on the basis it has to be said of very little data, to conjecture:

Conjecture 7.4. Let λ and θ be an arbitrary finite-dimensional selfcontragredient irreducible representation and the adjoint irreducible representation, respectively, of E_6. Then

$$\langle\lambda \otimes \{2\}, \theta\rangle = \begin{cases} \frac{1}{2}b(\lambda) & \text{if } b(\lambda) \text{ is even} \\ \frac{1}{2}(b(\lambda) - 1) & \text{if } b(\lambda) \text{ is odd} \end{cases}$$

$$\langle\lambda \otimes \{1^2\}, \theta\rangle = \begin{cases} \frac{1}{2}b(\lambda) & \text{if } b(\lambda) \text{ is even} \\ \frac{1}{2}(b(\lambda) + 1) & \text{if } b(\lambda) \text{ is odd.} \end{cases}$$

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